https://www.linkedin.com/feed/update/urn:li:activity:6528815998223683585

Problem1.(32-nd Russian Math. Olympiad 2005-2006, Final Round, 11-th form,1-st Day)

Prove that $\sin\sqrt{x} < \sqrt{\sin x}$ whenever $0 < x < \frac{\pi}{2}$.

Solution by Arkady Alt , San Jose ,California, USA.

Solution 1(with calculus)

For any positive t holds inequalities (Taylor's expansion for $\cos t$ and $\sin t$)

$\cos t > 1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720}$ and $\sin t > t - \frac{t^3}{6}$.

Let $0 < x < \frac{\pi}{2}$. Then $\sin\sqrt{x} < \sqrt{\sin x} \Leftrightarrow 2\sin^2\sqrt{x} < 2\sin x \Leftrightarrow$

$1 - \cos 2\sqrt{x} < 2\sin x$ and we have

$$1 - \cos 2\sqrt{x} < 1 - \left(1 - \frac{(2\sqrt{x})^2}{2} + \frac{(2\sqrt{x})^4}{24} - \frac{(2\sqrt{x})^6}{720}\right) \Leftrightarrow$$

$1 - \cos 2\sqrt{x} < 2x - \frac{2x^2}{3} + \frac{4x^3}{45}, \quad 2x - \frac{x^3}{3} < 2\sin x$.

Since $\left(2x - \frac{x^3}{3}\right) - \left(2x - \frac{2x^2}{3} + \frac{4x^3}{45}\right) = \frac{2x^2}{3} - \frac{x^3}{3} - \frac{4x^3}{45} =$

$\frac{2x^2}{3} - \frac{19x^3}{45} = \frac{x^2(30 - 19x)}{45}$ and $\frac{\pi}{2} < \frac{30}{19}$ then for $0 < x < \frac{\pi}{2}$

we obtain $1 - \cos 2\sqrt{x} < 2x - \frac{2x^2}{3} + \frac{4x^3}{45} < 2x - \frac{x^3}{3} < 2\sin x$.

Solution 2.(without calculus)

If $x = 1$ then $\sin 1 < 1 \Rightarrow \sqrt{\sin 1} > \sin 1 = \sin\sqrt{1}$.

Let $0 < x < 1$. Since $\sin x > x\cos x,\ x < \sqrt{x} \Rightarrow \cos x > \cos\sqrt{x}$,

$\cos\sqrt{x} > \cos^2\sqrt{x}$ (because $\cos\sqrt{x} \in (0,1)$) and $\sqrt{x} > \tan\sqrt{x}$

then $\sqrt{\sin x} > \sqrt{x}\cdot\cos\sqrt{x} > \tan\sqrt{x}\cdot\cos\sqrt{x} = \sin\sqrt{x}$.

And at last if $x \in (1, \pi/2)$ then $\sqrt{x} < x \Rightarrow \sin\sqrt{x} < \sin x$.

Since $0 < \sin x < 1 \Rightarrow \sin x < \sqrt{\sin x}$ we obtain $\sin\sqrt{x} < \sqrt{\sin x}$.